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A Note on the Filtered Least Squares Minimal Norm Solution of First Kind Equations

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A convergence theorem for J. W. Lee and P. M. Prenter's filtered least squares method for solving the Fredholm first kind equation $Kf = g$ is corrected. Under suitable restrictions the filtered least squares method is shown to be well posed under compact perturbations in K and arbitrary perturbations in g.

In [4] Lee and Prenter discuss the filtered LSMN (least squares minimal norm) solution of $Kf = g$, where K is a compact operator (continuous linear transformation) from complex Hilbert space H_1 into complex Hilbert space H_2, $f \in H_1$, and $g \in H_2$. His paper corrects Lee and Prenter's Theorem 5.2 [4, p. 11] and shows that in the corrected setting the filtered LSMN solution of $Kf = g$ is well posed with respect to compact perturbations in K and arbitrary perturbations in g.

Let $\{\lambda\}_{i \in I}$ denote the positive eigenvalues of the compact self-adjoint operator K^*K, written in decreasing order and listed according to multiplicity, where I can be at most countable. Let $\{u_i\}_{i \in I}$ be a set of distinct orthonormal eigenvectors of K^*K such that $K^*Ku_i = \lambda_i u_i$, for all $i \in I$. If $\mu_i = (\lambda_i)^{1/2}$, for each i, then define $v_i = Ku_i/\mu_i$. Then, $\{v_i\}_{i \in I}$ is also an orthonormal set. The collection $\{u_i, v_i; \mu_i\}_{i \in I}$ is called the *singular system of* K. Then for each $f \in H_1$, $Kf = \sum_{i \in I} \mu_i (f, u_i)\, v_i$, where (f, u_i) denotes the inner product of f and u_i [5, Theorem 1.9.3]. Let f_0 denote the least squares solution of minimum norm (Picard solution) of $Kf = g$ [4, p. 8]. Recall that f_0 exists if and only if $g \in R(K) \oplus R(K)^\perp$, where $R(K)$ denotes the range of K. Thus for $g \in R(K) \oplus R(K)^\perp$, $f_0 = \sum_{j \in I} (1/\mu_j)(g, v_j)\, u_j$.

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Let $\delta > 0$ and define $I_\delta = \{j \in I: \mu_j > \delta\}$. Observe that I_δ is a finite set. Define the *filtered LSMN solution*, f_δ, *of* $Kf = g$ to be $f_\delta = \sum_{j \in I_\delta} (1/\mu_j)(g, v_j)\, u_j$ as in [4].

Suppose $\{K_n\}$ is a sequence of compact operators such that $K_n: H_1 \to H_2$ and $\lim_{n\to\infty} \|K - K_n\| = 0$. We assume that, for all n, $g \in R(K_n) \oplus R(K_n)^\perp$. Observe that this condition will always hold if K_n has finite rank. If $\{u_j^n, v_j^n; \mu_j^n\}_{j \in I^n}$ is the singular system for K_n then the least squares solutions of minimum norm, f_0^n, of $K_n f = g$ is $f_0^n = \sum_j (1/\mu_j^n)(g, v_j^n)\, u_j^n$. Let f_δ^n denote the filtered LSMN solution of $K_n f = g$ and let $I_\delta^n = \{j \in I^n | \mu_j^n > \delta\}$. Then, $f_\delta^n = \sum_{j \in I_\delta^n} (1/\mu_j^n)(g, v_j^n)\, u_j^n$. Lee and Prenter [4, Theorem 5.2] state following theorem.

THEOREM (LEE-PRENTER). *Suppose* K, $K_n: H_1 \to H_2$ *are compact operators such that* $\lim_{n\to\infty} \|K - K_n\| = 0$. *Fix* $\delta > 0$ *and let* f_δ *and* f_δ^n *be the filtered LSMN solutions of* $Kf = g$ *and* $K_n f = g$, *respectively. Then,* $\lim_{n\to\infty} \|f_\delta^n - f_\delta\| = 0$.

The following example indicates that the above theorem does not hold for certain choices of $\delta > 0$ and that the value of $\lim_{n\to\infty} \|f_\delta^n - f_\delta\|$ can be large no matter how small $\delta > 0$ is chosen.

EXAMPLE. If $K: H_2 \to H_1$ is a compact operator with singular system $\{u_j, v_j; \mu_j\}_{j=1}^\infty$, then $Kf = \sum_j \mu_j (f, u_j)\, v_j$, for all $f \in H_1$. Suppose $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{N-1} > \mu_N > 0$. If we choose $\delta = \mu_N$, then $I_\delta = \{j | \mu_j > \delta\} = \{1,..., N-1\}$. For any $\varepsilon > 0$, we define $K_\varepsilon f = \sum_{j=1}^{N-1} \mu_j (f, u_j)\, v_j + (\mu_N + \varepsilon)(f, u_N)\, v_N + \sum_{j > N} \mu_j (f, u_j)\, v_j$, for $f \in H_1$. Choose $\varepsilon > 0$ so that $\mu_{N-1} > \mu_N + \varepsilon > \mu_{N+1}$. Then

$$\begin{aligned} \|K - K_\varepsilon\| &= \sup_{\|f\|=1} \|(K - K_\varepsilon) f\| \\ &= \sup_{\|f\|=1} \|\varepsilon (f, u_N)\, v_N\| \\ &= \sup_{\|f\|=1} \varepsilon \cdot |(f, u_N)| \cdot \|v_N\| \\ &\leqslant \sup_{\|f\|=1} \varepsilon \cdot \|f\| \cdot \|u_N\| \\ &= \varepsilon. \end{aligned}$$

Also, since $\mu_N + \varepsilon > \mu_N = \delta$, $I_\delta^\varepsilon = \{j | \mu_j^\varepsilon > \delta\} = \{1,..., N\}$. Let $g \in R(K) \oplus R(K)^\perp$. Since $R(K_\varepsilon) = R(K)$, $g \in R(K_\varepsilon) \oplus R(K_\varepsilon)^\perp$. Then $f_\delta = \sum_{j=1}^{N-1} (1/\mu_j)(g, v_j)\, u_j$ and $f_\delta^\varepsilon = \sum_{j=1}^{N-1} (1/\mu_j)(g, v_j)\, u_j + ((g, V_N)/(\mu_N + \varepsilon))\, u_N$. Thus, $\|f_\delta^\varepsilon - f_\delta\| = (\mu_N + \varepsilon)^{-1} |(g, v_N)| \cdot \|u_N\| = |(g, v_N)|/(\mu_N + \varepsilon)$. Therefore, $\lim_{\varepsilon\to 0} \|f_\delta^\varepsilon - f_\delta\| = |(g, v_N)|/\mu_N$, which is not necessarily zero. ∎

However, the above theorem by Lee and Prenter is true when one assumes that $\delta > 0$ is chosen so that δ is not equal to any of the singular values of K.

THEOREM. *Suppose $K, K_n: H_1 \to H_2$ are compact operators such that $\lim_{n\to\infty} \|K - K_n\| = 0$. Choose $\delta > 0$ such that $\delta \neq \mu_j$, for all j, where $\{\mu_j\}$ is the set of singular values of K. Then,* $\lim_{n\to\infty} \|f_\delta^n - f_\delta\| = 0$.

Let $B(H)$ denote the set of all continuous linear transformations from the complex Hilbert space H into H. If $L \in B(H)$ and if λ is an isolated point of the spectrum of L, $\sigma(L)$, then let $\Gamma = \{\lambda + \varepsilon e^{i\theta}: \theta \in [0, 2\pi]\}$, where $\varepsilon > 0$ is chosen small enough so that λ is the only point in $\sigma(L)$ that is inside or on Γ.

Define the *spectral projection* P *associated with* λ to be $P = -1/2\pi i \int_\Gamma (L - zI)^{-1}\, dz$. Kato [3, pp. 39, 178, 184] shows that $P^2 = P$, $PL = LP$ and, if $\Gamma^* = \{\bar{z}: z \in \Gamma\}$, then $P^* = -1/2\pi i \int_{\Gamma^*} (L^* - zI)^{-1}\, dz$. If $L^* = L$, then L has real spectrum. Hence, since Γ is a circle centered at the real number λ, $\Gamma^* = \Gamma$. Therefore $P^* = P$ and P is an orthogonal projection that commutes with L.

If λ is an eigenvalue of L, we define the *geometric multiplicity of* λ to be $\dim N(L - \lambda I)$, where $N(L - \lambda I)$ is the null space of $L - \lambda I$. If the eigenvalue λ is also an isolated point of $\sigma(L)$, we define the *algebraic multiplicity of* λ to be $\dim R(P)$, where P is the spectral projection associated with λ. Ringrose [5, pp. 51, 67] proves that, if λ is an eigenvalue of a compact self-adjoint operator L, then the algebraic multiplicity of λ equals the geometric multiplicity of λ and $R(P) = N(L - \lambda I)$, where P is the spectral projection associated with λ. Therefore P is the orthogonal projection onto $N(L - \lambda I)$. In the following theorem, we state two results contained in Kato [3, Theorem IV.3.16].

LEMMA. *Suppose L, $L_\varepsilon \in B(H)$ are compact self-adjoint operators such that $\|L - L_\varepsilon\| < \varepsilon$, λ is a non-zero eigenvalue of L with multiplicity m, and $U = \{z: |z - \lambda| < d\}$, where d is small enough so that $U \cap \sigma(L) = \{\lambda\}$. Then, for $\varepsilon > 0$ sufficiently small,*

(a) $\sigma(L_\varepsilon) \cap U = \{\lambda_1^\varepsilon, \lambda_2^\varepsilon, \dots, \lambda_r^\varepsilon\}$, *where λ_i^ε is an eigenvalue of finite multiplicity m_i^ε and $\sum_{i=1}^r m_i^\varepsilon = m$; and,*

(b) *if $P_{\lambda_i^\varepsilon}$ is the orthogonal projection onto $N(L_\varepsilon - \lambda_i^\varepsilon I)$ and $P_\varepsilon = \sum_{i=1}^r P_{\lambda_i^\varepsilon}$, then* $\lim_{\varepsilon\to 0} \|P_\varepsilon - P_\lambda\| = 0$, *where P_λ is the orthogonal projection onto $N(L - \lambda I)$.*

The idea behind the proof of this theorem is simple; namely, write $f_\delta = \sum_{\mu_i > \delta} ((g, v_i)/\mu_i)\, u_i$ and $f_\delta^n = \sum_{\mu_i^n > \delta} ((g, v_i^n)/\mu_i^n)\, u_i^n$ and then show $\mu_i^n \to \mu_i$, $v_i^n \to v_i$, and $u_i^n \to u_i$. From part (a) of the above lemma it follows that $\mu_i^n \to \mu_i$. However to prove $v_i^n \to v_i$ and $u_i^n \to u_i$ one must pick the eigenvectors u_i and u_i^n very carefully, otherwise it will not necessarily be true. This difficulty is compounded when one allows for eigenvalues of various multiplicities. Because of this, it turns out to be easier to look at the orthogonal projections onto the associated eigenspaces and show that these have the desired convergence properties.

Proof of Theorem. Since $\delta \neq \mu_j$, for all j, there exists N such that $\mu_N > \delta > \mu_{N+1}$. Thus, $I_\delta = \{1,..., N\}$. From the definition of the singular system $\{u_i, v_i; \mu_i\}$ of K, we recall that, if $\lambda_i = \mu_i^2$, then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots > 0$ is the set of positive eigenvalues of K^*K with corresponding orthonormal eigenvectors $\{u_i\}$. The set $\{v_i\}$ is obtained by setting $v_i = (1/\mu_i) Ku_i$.

Let $\gamma_1 > \gamma_2 > \cdots > \gamma_t > 0$ be the set of distinct eigenvalues of K^*K in $\{\lambda_j : j \in I_\delta\}$. If m_i is the multiplicity of γ_i, then $\sum_{i=1}^t m_i = N$. If we rename the set of orthonormal eigenvectors corresponding to γ_i to be $\{u_j^i\}_{j=1}^{m_i}$, then $N_i \equiv N(K^*K - \gamma_i I) = \operatorname{span}\{u_1^i, u_2^i,..., u_{m_i}^i\}$. Define $v_j^i = Ku_j^i/(\gamma_i)^{1/2}$ and denote the orthogonal projection of H_1 onto N_i by P_i.

Choose d so that $0 < d < \frac{1}{2}\min[\{\gamma_i - \gamma_{i+1}\}_{i=1}^{t-1} \cup \{\gamma_t - \delta^2, \delta^2\}]$, and let $U_i^d = \{z : |z - \gamma_i| < d\}$. By part (a) of the above lemma (with $L = K^*K$), for large enough n, we have $\sigma(K_n^* K_n) \cap U_i^d = \{\lambda_{ij}^n j \in I_i^n\}$, where λ_{ij}^n is an eigenvalue of $K_n^* K_n$ with finite multiplicity m_{ij}^n, I_i^n is a finite set, and $\sum_{j \in I_i^n} m_{ij}^n = m_i$. If $N_{ij}^n = N(K_n^* K_n - \lambda_{ij}^n I)$, then $N_{ij}^n = \operatorname{span}\{u_{ijk}^n\}_{k=1}^{m_{ij}^n}$, where $\{u_{ijk}^n : 1 \leqslant k \leqslant m_{ij}^n\}$ is an orthonormal set of eigenvectors of $K_n^* K_n$ corresponding to λ_{ij}^n. Let P_{ij}^n denote the orthogonal projection of H_1 onto N_{ij}^n and define $P_i^n = \sum_{j \in I_i^n} P_{ij}^n$. Define $\mu_{ij}^n = (\lambda_{ij}^n)^{1/2}$ and define $v_{ijk}^n = K_n u_{ijk}^n / \mu_{ij}^n$. By our choice of d, $\mu_{ij}^n = (\lambda_{ij}^n)^{1/2} > \delta$, for $j \in I_i^n$ and $1 \leqslant i \leqslant t$. Hence, if $\mu_1^n \geqslant \mu_2^n \geqslant \cdots \geqslant \mu_n^n \geqslant \cdots > 0$ are the singular values of K_n listed according to multiplicity and $I_\delta^n = \{j | \mu_j^n > \delta\}$, then $I_\delta^n = \{1,..., N\}$.

Recalling that $0 < d < \frac{1}{2}\min[\{\gamma_i - \gamma_{i+1}\}_{i=1}^{t-1} \cup \{\gamma_t - \delta^2, \delta^2\}]$, an elementary calculation shows that $1/\lambda_{ij}^n = 1/\gamma_i + \varepsilon_{ij}^n$, where $|\varepsilon_{ij}^n| \leqslant 2d/\gamma_i^2$. Using this fact, we write f_δ and f_δ^n in a form to which we can apply the above lemma.

$$\begin{aligned}
f_\delta &= \sum_{i=1}^{t} (\gamma_i)^{-1/2} \sum_{j=1}^{m_i} (g, v_j^i)\, u_j^i \\
&= \sum_{i=1}^{t} (\gamma_i)^{-1/2} \sum_{j=1}^{m_i} (g, (\gamma_i)^{-1/2} K u_j^i)\, u_j^i \\
&= \sum_{i=1}^{t} \frac{1}{\gamma_i} \sum_{j=1}^{m_i} (K^* g, u_j^i)\, u_j^i \\
&= \sum_{i=1}^{t} \frac{1}{\gamma_i} P_i(K^* g).
\end{aligned}$$

$$\begin{aligned}
f_\delta^n &= \sum_{i=1}^{t} \sum_{j\in I_i^n} (\lambda_{ij}^n)^{-1/2} \sum_{k=1}^{m_{ij}^n} (g, v_{ijk}^n)\, u_{ijk}^n \\
&= \sum_{i=1}^{t} \sum_{j\in I_i^n} (\lambda_{ij}^n)^{-1/2} \sum_{k=1}^{m_{ij}^n} (g, (\lambda_{ij}^n)^{-1/2} K_n u_{ijk}^n)\, u_{ijk}^n \\
&= \sum_{i=1}^{t} \sum_{j\in I_i^n} \frac{1}{\lambda_{ij}^n} \sum_{k=1}^{m_{ij}^n} (K_n^* g, u_{ijk}^n)\, u_{ijk}^n \\
&= \sum_{i=1}^{t} \sum_{j\in I_i^n} \frac{1}{\lambda_{ij}^n} P_{ij}^n (K_n^* g) \\
&= \sum_{i=1}^{t} \sum_{j\in I_i^n} \left(\frac{1}{\gamma_i} + \varepsilon_{ij}^n\right) P_{ij}^n (K_n^* g) \\
&= \sum_{i=1}^{t} \sum_{j\in I_i^n} \frac{1}{\gamma_i} P_{ij}^n(K_n^* g) + \sum_{i=1}^{t} \sum_{j\in I_i^n} \varepsilon_{ij}^n P_{ij}^n(K_n^* g) \\
&= \sum_{i=1}^{t} \frac{1}{\gamma_i} \sum_{j\in I_i^n} P_{ij}^n(K_n^* g) + \sum_{i=1}^{t} \sum_{j\in I_i^n} \varepsilon_{ij}^n P_{ij}^n(K_n^* g) \\
&= \sum_{i=1}^{t} \frac{1}{\gamma_i} P_i^n(K_n^* g) + \sum_{i=1}^{t} \sum_{j\in I_i^n} \varepsilon_{ij}^n P_{ij}^n(K_n^* g).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|f_\delta^n - f_\delta\| &= \left\| \sum_{i=1}^{t} \frac{1}{\gamma_i} P_i^n(K_n^* g) + \sum_{i=1}^{t} \sum_{j\in I_i^n} \varepsilon_{ij}^n P_{ij}^n(K_n^* g) \right. \\
&\qquad \left. - \sum_{i=1}^{t} \frac{1}{\gamma_i} P_i(K_n^* g) \right\| \\
&\leqslant \left\| \sum_{i=1}^{t} \frac{1}{\gamma_i} (P_i^n(K_n^* g) - P_i(K^* g)) \right\| \\
&\qquad + \left\| \sum_{i=1}^{t} \sum_{j\in I_i^n} \varepsilon_{ij}^n P_{ij}^n(K_n^* g) \right\| \\
&\leqslant \sum_{i=1}^{t} \frac{1}{\gamma_i} \|P_i^n(K_n^* g) - P_i(K_n^* g)\| \\
&\qquad + \sum_{i=1}^{t} \sum_{j\in I_i^n} |\varepsilon_{ij}^n| \cdot \|P_{ij}^n(K_n^* g)\|.
\end{aligned}$$

Next we obtain estimates on each of these two sums separately.

$$\sum_{i=1}^{t} \frac{1}{\gamma_i} \|P_i^n(K_n^* g) - P_i(K^*g)\|$$

$$\leqslant \sum_{i=1}^{t} \frac{1}{\gamma_i} \{\|P_i^n(K_n^* g) - P_i^n(K^*g)\| + \|P_i^n(K^*g) - P_i(K^*g)\|\}$$

$$\leqslant \sum_{i=1}^{t} \frac{1}{\gamma_i} \cdot \|P_i^n\| \cdot \|K_n^* g - K^*g\| + \sum_{i=1}^{t} \frac{1}{\gamma_i} \cdot \|P_i^n - P_i\| \cdot \|K^*g\|$$

$$\leqslant \sum_{i=1}^{t} \frac{1}{\gamma_i} \cdot \|K_n^* - K^*\| \cdot \|g\| + \sum_{i=1}^{t} \frac{1}{\gamma_i} \cdot \|P_i^n - P_i\| \cdot \|K^*\| \cdot \|g\|$$

$$\leqslant \|g\| \cdot \|K_n - K\| \cdot \sum_{i=1}^{t} \frac{1}{\gamma_i} + \|g\| \cdot \|K\| \cdot \sum_{i=1}^{t} \frac{1}{\gamma_i} \cdot \|P_i^n - P_i\|,$$

and

$$\sum_{i=1}^{t} \sum_{j \in I_i^n} |\varepsilon_{ij}^n| \cdot \| P_{ij}^n(K_n^* g)\|$$

$$\leqslant \sum_{i=1}^{t} \sum_{j \in I_i^n} 2d/\gamma_i^2 \cdot \|P_{ij}^n\| \cdot \|K_n^*\| \cdot \|g\|$$

$$= \sum_{i=1}^{t} \sum_{j \in I_i^n} 2d/\gamma_i^2 \cdot \|K_n\| \cdot \|g\|$$

$$\leqslant 2d \cdot \|K_n\| \cdot \|g\| \cdot \sum_{i=1}^{t} m_i/\gamma_i^2.$$

Combining the above two estimates, we have

$$\|f_\delta^n - f_\delta\| \leqslant \|g\| \left\{ \|K_n - K\| \cdot \sum_{i=1}^{t} \frac{1}{\gamma_i} + \|K\| \cdot \sum_{i=1}^{t} \frac{1}{\gamma_i} \|P_i^n - P_i\| \right.$$

$$\left. + 2d \cdot \|K_n\| \cdot \sum_{i=1}^{t} \frac{m_i}{\gamma_i^2} \right\}.$$

By part (b) of the above lemma (with $L = K^*K$), $\lim_{n\to\infty} \|P_i^n - P_i\| = 0$, for each i. Therefore, since $\lim_{n\to\infty} \|K_n - K\| = 0$ and since d can be chosen arbitrarily small, $\lim_{n\to\infty} \|f_\delta^n - f_\delta\| = 0$. ∎

COROLLARY. *If δ is chosen as in the above theorem, then the filtered LSMN solution of $KF = g$ is well posed under compact perturbations in K and arbitrary perturbations in g.*

Proof. Let $\varepsilon > 0$ and assume that $\|K - K_\varepsilon\| < \varepsilon$ and $\| g - g_\varepsilon\| < \varepsilon$. Let $f_\delta(g)$ and $f_\delta^\varepsilon(g)$ denote the filtered LSMN solutions of $Kf = g$ and $K_\varepsilon f = g$, respectively. Recall that $f_\delta(g) = \sum_{j\in I_\delta} (1/\mu_j)(g, v_j)\, u_j$. Therefore,

$$\begin{aligned}
\|f_\delta(g_\varepsilon) - f_\delta(g)\| &= \left\| \sum_{j\in I_\delta} \frac{1}{\mu_j} (g_\varepsilon, v_j)\, u_j - \sum_{j\in I_\delta} \frac{1}{\mu_j} (g, v_j)\, u_j \right\| \\
&= \left\| \sum_{j\in I_\delta} \frac{1}{\mu_j} (g_\varepsilon - g, v_j)\, u_j \right\| \\
&\leqslant \sum_{j\in I_\delta} \frac{1}{\mu_j} |(g_\varepsilon - g, v_j)| \cdot \|u_j\| \\
&\leqslant \sum_{j\in I_\delta} \frac{1}{\mu_j} \cdot \| g_\varepsilon - g\| \cdot \|v_j\| \\
&< \varepsilon \cdot \sum_{j\in I_\delta} \frac{1}{\mu_j}.
\end{aligned}$$

Hence, $\lim_{\varepsilon\to 0^+} \|f_\delta(g_\varepsilon) - f_\delta(g)\| = 0$.

By the above theorem, we know that $\lim_{\varepsilon\to 0^+} \|f_\delta^\varepsilon(g_\varepsilon) - f_\delta(g_\varepsilon)\| = 0$. Hence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \|f_\delta^\varepsilon(g_\varepsilon) - f_\delta(g)\| \\
&\qquad \leqslant \lim_{\varepsilon\to 0^+} \{\|f_\delta^\varepsilon(g_\varepsilon) - f_\delta(g_\varepsilon)\| + \|f_\delta(g_\varepsilon) - f_\delta(g)\|\} = 0. \quad \blacksquare
\end{aligned}$$

For further discussion of the filtered LSMN solution of $Kf = g$ see [1, 2, 6].

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